Definition 7.11 (Gamma-function):

For $x > 0$ define

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$$

Remark 7.8:

This improper integral converges as

i) $t^{x-1} e^{-t} \le \frac{1}{t^{1-x}}$ for all $t > 0$,

ii) $t^{x-1} e^{-t} \le \frac{1}{t^2}$ for $t \geq t_0$,

as $\lim_{t\to\infty} t^{x+1} e^{-t} = 0$

$\Rightarrow$ one can write

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt = \int_0^{t_0} t^{x-1} e^{-t} dt + \int_{t_0}^\infty t^{x-1} e^{-t} dt$$

$$\le \int_0^{t_0} \frac{1}{t^{1-x}} dt + \int_{t_0}^\infty \frac{1}{t^2} dt$$

$$\overset{t = t_0 \tilde{t}}{=} t_0^x \underbrace{\int_0^1 \frac{1}{\tilde{t}^{1-x}} d\tilde{t}}_{\substack{\text{convergent} \\ \text{(see Ex. 7.11)}}} + \frac{1}{t_0} \underbrace{\int_1^\infty \frac{1}{\tilde{t}^2} d\tilde{t}}_{\substack{\text{convergent} \\ \text{(see Ex. 7.10)}}}$$

Prop. 7.17:

We have $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$ and

$$x\Gamma(x) = \Gamma(x+1) \quad \text{for all } x \in \mathbb{R}_{>0}.$$

Proof:

Partial integration gives

$$\int_{\varepsilon}^{R} t^x e^{-t}\,dt = -t^x e^{-t}\Big|_{t=\varepsilon}^{t=R} + x\int_{\varepsilon}^{R} t^{x-1}e^{-t}\,dt$$

$$\Rightarrow \lim_{\substack{R\to\infty,\\ \varepsilon\to 0}} \int_{\varepsilon}^{R} t^x e^{-t}\,dt = \Gamma(x+1)$$

$$= \lim_{\substack{R\to\infty,\\ \varepsilon\to 0}} \left(-t^x e^{-t}\Big|_{t=\varepsilon}^{t=R}\right) + \lim_{\substack{R\to\infty,\\ \varepsilon\to 0}} \left(x\int_{\varepsilon}^{R} t^{x-1}e^{-t}\,dt\right)$$

$$= 0 + x\Gamma(x).$$

As

$$\Gamma(1) = \lim_{R\to\infty}\int_0^R e^{-t}\,dt = \lim_{R\to\infty}\left(1-e^{-R}\right) = 1,$$

it follows that

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\cdot\ldots\cdot 1\cdot\Gamma(1) = n!$$

$\square$

Remark 7.9:

The function $\Gamma: \mathbb{R}_{>0} \to \mathbb{R}$ interpolates the factorial (which is only defined for natural numbers). This does not show the uniqueness of the Gamma-function, yet. For this we need the following:

Definition 7.12:

Let $I \subset \mathbb{R}$ be an interval. A positiv function $F: I \longrightarrow \mathbb{R}_{>0}$ is called "logarithmic convex", if the function $\log F: I \longrightarrow \mathbb{R}$ is convex.

$\Longleftrightarrow$ F is logarithmically convex if and only if $\forall\ x, y \in I$ and $0 < \lambda < 1$:

$$F(\lambda x + (1-\lambda) y) \leq F(x)^{\lambda} F(y)^{1-\lambda}.$$

Proposition 7.18:

The function $\Gamma: \mathbb{R}_{>0} \to \mathbb{R}$ is logarithmically convex. (Omit the proof here)

Note: Prop. 7.18 shows that Γ is continuous on $\mathbb{R}_{>0}$ as it is a convex function on an open interval.

Proposition 7.19:

Let $F: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a function with the following properties:

i) $F(1) = 1$,

ii) $F(x+1) = x F(x) \quad \forall \; x \in \mathbb{R}_{>0}$,

iii) F is logarithmically convex.

Then $F(x) = \Gamma(x) \quad \forall \; x \in \mathbb{R}_{>0}$.

Proof:

As the Gamma-function has properties i)–iii), it suffices to prove that any function F with properties i) – iii) is unique.

ii) $\Longrightarrow \quad F(x+n) = F(x)\,x\,(x+1)\cdot \ldots \cdot (x+n-1)$

for all $x > 0$ and all $n \in \mathbb{N}$, $n \geq 1$.

In particular: $F(n+1) = n! \quad \forall \; n \in \mathbb{N}$.

$\Longrightarrow$ it suffices to show uniquenes for $0 < x < 1$

As $n + x = (1-x)n + x(n+1)$, it follows from iii) that

$$F(x+n) \leq F(n)^{1-x} F(n+1)^{x} = F(n)^{1-x} F(n)^{x} n^{x} = (n-1)!\, n^{x}$$

From $n+1 = x(n+x) + (1-x)(n+1+x)$ it follows

$$n! = F(n+1) \leq F(n+x)^x F(n+1+x)^{1-x} = F(n+x)(n+x)^{1-x}$$

Combining the two inequalities, one obtains

$$n!(n+x)^{x-1} \leq F(n+x) \leq (n-1)!\, n^x$$

and further

$$a_n(x) := \frac{n!(n+x)^{x-1}}{x(x+1)\cdot \ldots \cdot (x+n-1)} \leq F(x)$$

$$\leq \frac{(n-1)!\, n^x}{x(x+1)\cdot \ldots \cdot (x+n-1)} =: b_n(x)$$

As $\frac{b_n(x)}{a_n(x)} = \frac{(n+x)n^x}{n(n+x)^x} \longrightarrow 1 \ (n \to \infty)$, we get

$$F(x) = \lim_{n\to\infty} \frac{(n-1)!\, n^x}{x(x+1)\cdot \ldots \cdot (x+n-1)}, \qquad (*)$$

thus F is uniquely determined. $\square$

Proposition 7.20:

For all $0 < x$ we have

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\cdot \ldots \cdot (x+n)}.$$

Proof:

As $\lim\limits_{n\to\infty} \frac{n}{x+n} = 1$, the claim follows for $0<x<1$ from the form $(*)$ obtained in the previous proof.

Furthermore,

$$\Gamma(y) = \Gamma(x+1) = x\Gamma(x) = \lim_{n\to\infty} \frac{n!\,n^x}{(x+1)\cdot\ldots\cdot(x+n)}$$

$$= \lim_{n\to\infty} \frac{n!\,n^{y-1}}{y(y+1)\cdot\ldots\cdot(y+n-1)}$$

$$= \lim_{n\to\infty} \frac{n!\,n^{y}}{y(y+1)\cdot\ldots\cdot(y+n-1)(y+n)} .$$

$\square$

<u>Corollary 7.2:</u>

For all $x > 0$ we have

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n},$$

where γ is the Euler-Mascheroni constant.

<u>Proof:</u>

From Prop. 7.20 it follows

$$\frac{1}{\Gamma(x)} = x \lim_{N\to\infty} \frac{(x+1)\cdot\ldots\cdot(x+N)}{N!} N^{-x}$$

$$= x \lim_{N\to\infty} \left(1 + \frac{x}{1}\right)\cdot\ldots\cdot\left(1+\frac{x}{N}\right)\exp(-x\log N)$$

$$= x \lim_{N\to\infty} \left(\prod_{n=1}^{N}\left(1+\frac{x}{n}\right)e^{-x/n}\right)\exp\left(\sum_{n=1}^{N}\frac{x}{n} - x\log N\right)$$

As $\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{x}{N} - x\log N\right) = x\gamma$, one obtains

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1+\frac{x}{n}\right) e^{-x/n}$$

$\square$

Example 7.14:

i) We show as an application of Cor. 7.2 that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof:

We can represent $\Gamma\left(\frac{1}{2}\right)$ in two different ways:

$$\Gamma\left(\tfrac{1}{2}\right) = \lim_{n\to\infty} \frac{n!\sqrt{n}}{\frac{1}{2}\left(1+\frac{1}{2}\right)\left(2+\frac{1}{2}\right)\cdot\ldots\cdot\left(n+\frac{1}{2}\right)},$$

$$\Gamma\left(\tfrac{1}{2}\right) = \lim_{n\to\infty} \frac{n!\sqrt{n}}{\left(1-\frac{1}{2}\right)\left(2-\frac{1}{2}\right)\cdot\ldots\cdot\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}$$

Multiplication gives

$$\Gamma\left(\tfrac{1}{2}\right)^2 = \lim_{n\to\infty} \frac{2n}{n+\frac{1}{2}} \cdot \frac{(n!)^2}{\left(1-\frac{1}{4}\right)\left(4-\frac{1}{4}\right)\cdot\ldots\cdot\left(n^2-\frac{1}{4}\right)}$$

$$= 2\lim_{n\to\infty} \prod_{k=1}^{n} \frac{k^2}{k^2-\frac{1}{4}} = \pi,$$

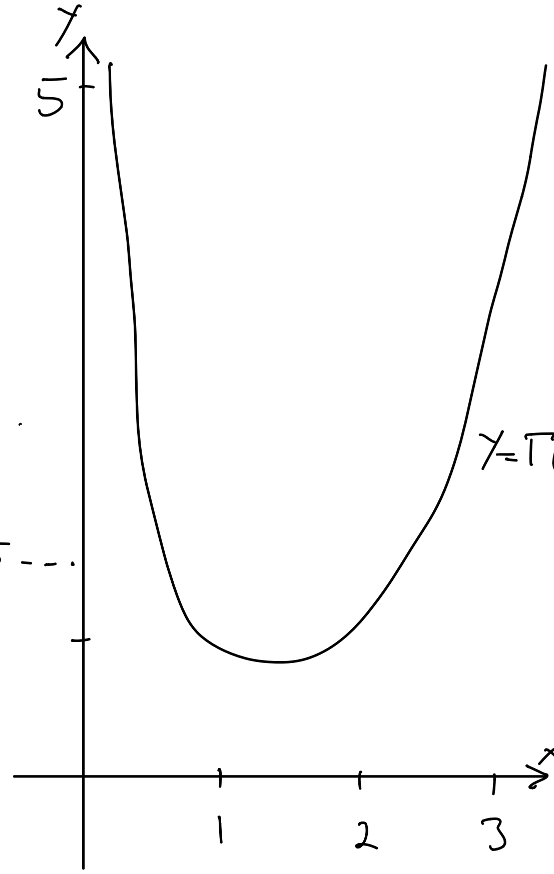
where in the last step we used Wallis' product representation for π. Altogether: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\square$

Using $(n+\frac{1}{2})\Gamma(n+\frac{1}{2}) = \Gamma(n+1+\frac{1}{2})$,

we can obtain from this all values of $\Gamma(n+\frac{1}{2})$ for all n:

x	$\Gamma(x)$
0.5	$\sqrt{\pi} = 1.77245\ldots$
1	$0! = 1$
1.5	$\frac{1}{2}\sqrt{\pi} = 0.88622\ldots$
2	$1! = 1$
2.5	$\frac{3}{4}\sqrt{\pi} = 1.32934\ldots$
3	$2! = 2$
3.5	$\frac{15}{8}\sqrt{\pi} = 3.32335\ldots$
4	$3! = 6$



From

$$\lim_{x \searrow 0} x\Gamma(x) = \lim_{x \searrow 0} \Gamma(x+1) = \Gamma(1) = 1,$$

it follows that $\Gamma(x)$ behaves asymptotically as $\frac{1}{x}$ for $x \to 0$.

ii) $$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Proof:

The substitution $x = t^{1/2}$, $dx = \frac{1}{2} t^{-1/2}\,dt$ gives

$$\int_{\varepsilon}^{R} e^{-x^2}\,dx = \frac{1}{2}\int_{\varepsilon^2}^{R^2} t^{-1/2} e^{-t}\,dt,$$

thus the limit $\varepsilon \to 0$, $R \to \infty$ gives

$$\int_{0}^{\infty} e^{-x^2}\,dx = \frac{1}{2}\int_{0}^{\infty} t^{-1/2} e^{-t}\,dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.$$

$\square$

Proposition 7.21 (Stirling):

The factorial has the asymptotic form

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

(where by "asymptotic" we mean here

$$\lim_{n\to\infty}\left(\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}\right) = 1.$$ It is not suggested or required here that the two sequences converge, only their ratio converges.)

Proof:

We define a function $\varphi : \mathbb{R} \to \mathbb{R}$ as follows

$$\varphi(x) := \frac{1}{2} x(1-x) \quad \text{for } x \in [0,1],$$

$$\varphi(x+n) := \varphi(x) \quad \text{for all } n \in \mathbb{Z} \text{ and } x \in [0,1].$$

From the trapeze law (Prop. 7.15) we obtain

$$\int_k^{k+1} \log x \, dx = \frac{1}{2}\big(\log(k) + \log(k+1)\big) + \int_k^{k+1} \frac{\varphi(x)}{x^2} dx$$

$$\left(\log''(x) = -\frac{1}{x^2} \right)$$

Summation over $k = 1, \ldots, n-1$ gives

$$\int_1^n \log x \, dx = \sum_{k=1}^n \log k - \frac{1}{2} \log n + \int_1^n \frac{\varphi(x)}{x^2} dx.$$

As $\int_1^n \log x \, dx = n \log n - n + 1$, it follows

$$\sum_{k=1}^n \log k = \left(n + \frac{1}{2}\right) \log n - n + a_n,$$

where

$$a_n := 1 - \int_1^n \frac{\varphi(x)}{x^2} dx.$$

Exponentiating both sides, we obtain ($c_n = e^{a_n}$)

$$(*) \quad n! = n^{n+\frac{1}{2}} e^{-n} c_n, \quad \text{so } c_n = \frac{n!}{\sqrt{n}} \frac{e^n}{n^n}.$$

As φ is bounded and $\int_1^\infty \frac{1}{x^2}\,dx < \infty$, the limit

$$a := \lim_{n \to \infty} a_n = 1 - \int_1^\infty \frac{\varphi(x)}{x^2}\,dx$$

exists and therefore also the limit

$$c := \lim_{n \to \infty} c_n = e^a.$$

We have

$$\frac{c_n^2}{c_{2n}} = \frac{(n!)^2 \sqrt{2n}\,(2n)^{2n}}{n^{2n+1}\,(2n)!} = \sqrt{2}\,\frac{2^{2n}(n!)^2}{\sqrt{n}\,(2n)!}$$

and $\lim_{n \to \infty} \frac{c_n^2}{c_{2n}} = \frac{c^2}{c} = c$. In order to compute c, we use Wallis' product representation

$$\pi = 2 \prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = 2 \lim_{n \to \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot \ldots \cdot 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)(2n+1)}$$

$$\Rightarrow \left(2 \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \right)^{1/2} = \sqrt{2}\, \frac{2 \cdot 4 \cdot \ldots \cdot 2n}{3 \cdot 5 \cdot \ldots \cdot (2n-1)\sqrt{2n+1}}$$

$$= \frac{1}{\sqrt{n+\frac{1}{2}}} \cdot \frac{2^2 \cdot 4^2 \cdot \ldots \cdot (2n)^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot (2n-1) \cdot 2n}$$

$$= \frac{1}{\sqrt{n+\frac{1}{2}}} \cdot \frac{2^{2n}(n!)^2}{(2n)!},$$

therefore
$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n}(n!)^2}{\sqrt{n}\,(2n)!}.$$

$$\Rightarrow\ C = \sqrt{2\pi}, \quad \text{or} \quad \lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\cdot n^n e^{-n}} = 1$$

$\square$

<u>Note</u> (error estimate):
$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}$$